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REVIEWS.

MATHEMATICAL PHILOSOPHY AND PHILOSOPHY OF MATHEMATICS.

Introduction to Mathematical Philosophy. By BERTRAND RUSSELL. New York, The Macmillan Company, 1919. 8 + 208 pages. Price $3.00.

The title of this book does not sufficiently indicate, perhaps no brief title could indicate clearly, the nature of the work. It is not so much an introduction to mathematical philosophy as to the philosophy of mathematics. The two 'philosophies' are very different things, the former being identified mainly by its method, the latter by its matter. An example of mathematical philosophy is the famous attempt of Spinoza to geometrize the philosophy of Descartes or his yet more famous attempt to construct the *Ethics* after the manner of Euclid. On the other hand, the philosophy of mathematics, as Mr. Russell himself here and elsewhere avows, is primarily concerned with such unanswered questions as arise in reflecting upon the nature of mathematics and the character of its foundations. This distinction being understood, we may say that there are two respects in which the present book may be properly said to be an introduction to the philosophy of mathematics. These respects may be indicated briefly as follows. When a question belonging to the philosophy of mathematics is at length acceptably answered, it ceases to be a question; the matter passes from the domain of philosophy to that of science, it becomes a genuine part of mathematics itself. Students of the logical foundations of mathematics are well aware, though the average professional philosopher is slow to learn, that this transition from speculation to knowledge has been a common and striking phenomenon in recent times. Many questions regarding the nature of such fundamentals as class, relation, number, continuity, infinity, and kindred matters, have been answered satisfactorily. The answers together with the ways of discovering them constitute the most fundamental branch of mathematical science. In the book before us Mr. Russell has endeavored to give the reader a stimulating acquaintance with some of these scientific results and methods. In this respect, then, the book is an introduction to the philosophy of mathematics, for the best approach to questions that have not been answered lies through acquaintance with the answers to those that have been answered. But the author has done more than that. Not only is the reader led in various directions *towards* the frontier of knowledge in this field but in several directions he is led *to* the frontier and is made to feel the challenge of unanswered questions that arise there. What is the characteristic—the necessary and sufficient criterion—of a mathematical proposition? What is to be the ultimate form of the theory of "logical types"? Is the multiplicative axiom, which is equivalent to that of Zermelo, true? In other words, is it true that, given any class of mutually exclusive classes, of which none is empty, there is at least one class having exactly one term in common with each of the given classes? Is every infinite class reflexive? Such are some of the frontier questions which Mr. Russell's introduction makes the reader aware of.

The scope and content of the book are fairly indicated by the titles of its 18 chapters, which are as follows:

(1) The series of natural numbers; (2) Definition of number; (3) Finitude and mathematical induction; (4) The definition of order; (5) Kinds of relations; (6) Similarity of relations; (7) Rational, real and complex numbers; (8) Infinite cardinal numbers; (9) Infinite series and ordinals; (10) Limits and continuity; (11) Limits and continuity of functions; (12) Selections and the multiplicative axiom; (13) The axiom of infinity and logical types; (14) Incompatibility and the theory of deduction; (15) Propositional functions; (16) Descriptions; (17) Classes; (18) Mathematics and logic.

In the space allotted to this review it is impossible to present a digest of these chapters, for each of these is itself only a semi-popular digest, in some cases only a very compact and somewhat meager digest, of the more elaborate and more systematic and technical handling of the same topics in Whitehead and Russell's *Principia Mathematica* and in Russell's earlier work, *The Principles of Mathematics*. Accordingly I shall not attempt to offer a digest of these various digests but shall content myself with giving a sketchy account of some of them in the hope that the reader may be thereby drawn to the book itself and by it to the mentioned *Principia* and the great spiritual enterprise it represents.

The supreme thesis of the book is that logic and mathematics, rightly understood, are not two distinct sciences but that they together literally constitute one science without any inner division or breach from the most primitive things in what is traditionally called logic to the latest developments in what is traditionally called mathematics. Why, then, does Mr. Russell open his discussion with a consideration of the natural numbers, or ordinary integers? Are these assumed to be the most primitive of known things in logic (*i.e.* mathematics)? Far from it. In the *Principia* one must march through more than 350 large pages of solid developments before the concept of even the first of the integers is encountered. In an introduction designed to attract instead of repelling the uninitiated it was necessary to take most of those developments for granted; it was necessary to begin with something familiar; and nothing is more familiar than the so-called natural numbers, notwithstanding the fact that very few people, even among mathematicians, have taken the pains to ascertain the significance of these numbers in terms of logical primitives.

Mr. Russell's discussion of the natural numbers attaches itself to the great, albeit now superseded, work of Peano. "Having reduced all traditional pure mathematics to the theory of the natural numbers, the next great step in logical analysis was to reduce this theory itself to the smallest set of premises and undefined terms from which it could be derived. This work was accomplished by Peano. He showed that the entire theory of the natural numbers could be derived from three primitive ideas and five primitive propositions in addition to those of pure logic." Having paid this deserved tribute to the pioneer work of the distinguished Italian, Mr. Russell proceeds to show in detail that and how Peano's three primitive (undefined) ideas—zero, number, successor—may be defined perfectly in purely logical terms and, moreover, that and how his five primitive (undemonstrated) propositions admit of rigorous demonstration by means of purely logical concepts and propositions. This beautiful achievement, accomplished in less than 30 pages, is very weighty as showing how the entire superstructure of traditional mathematics rests upon a purely logical basis.

The natural numbers are cardinal numbers. On the threshold of this subject three questions present themselves: (1) What is meant by saying that two given classes have the same cardinal number? (2) What is meant by the cardinal number of a given class? (3) What is a cardinal number? Two classes are *similar* if they can be paired in one–one fashion like the class whose members are the legs and the class whose members are the arms of a normal human body. The answer to question (1) is: two classes have the same cardinal number if they are similar. The answer to (2) is: the cardinal number of a given class c is the class C of all classes similar to c. *i.e.*; of all classes each of which has the same cardinal number as c. The answer to (3) is: the class C' of all classes C such that each of these has for its members all the classes c similar to one of them. It is interesting to note, in passing, how far the concept, cardinal number, is removed from the domain of perception.

Evidently cardinal numbers are signless, neither positive nor negative. In order to prepare the reader for definitions (in terms of logical constants) of positive and negative integers, rational numbers, real numbers, complex numbers and other sorts, it is necessary to acquaint him with a certain serviceable stock of relations and their chief properties—relations between individuals and classes, relations subsisting between classes, and relations, such as likeness of structure, holding between relations themselves. After such preparation, which the reader receives with pleasure and a growing sense of enlightenment, he readily discerns what manner of things the various kinds of number really are. He finds, for example, that a positive or a negative integer, instead of being, like a cardinal, a class of classes, is a specific kind of *relation* between such classes of classes. More precisely, if n be any cardinal (finite or infinite) and m be a finite cardinal, the positive integer, $+ m$, is the relation of the cardinal, $n + m$, to n, and the negative integer, $- m$, is the converse relation, namely that of n to $n + m$.

What is meant by a rational number? This, too, is a relation between cardinals but a different sort of relation from the foregoing one. The direct logical approach is through the notion of propositional function. A propositional function is any statement involving one or more variables. It has the form of a proposition but is not one, being neither true nor false since the variables in it are but blanks or empty symbols. If it contain but one variable, the function determines a class of terms, namely that composed of all the terms which, if substituted for the variable, yield true propositions. If it contains two variables, say x and y, it determines a dyadic relation, which for mathematics is a class of couples of terms, where a and b (taken in the order named) will be a couple, a member of the class, a constituent of the relation, if and only if, on being substituted, a for x and b for y, a true proposition results. We can now define precisely what is meant by the rational number m/n where m and n are definite finite cardinals. Let x and y be variables denoting finite cardinals. Then the statement $xn = ym$ is a propositional function determining a dyadic relation. The relation is a certain class of couples (a, b) of cardinals such that, if a and b be substituted respectively for x and y in the function, a true proposition results. This relation, this class of couples, is, by definition, the rational number or fraction m/n.

Such rationals are, like cardinals, signless. Positive and negative fractions are relations between the signless fractions and hence are relations of relations of cardinals.

What is a real number? There is room here for only a hint. May it suffice. A real number is neither a class of classes nor a relation of any kind used above. It is, however, a kind of relation called a *segment* of rationals. What the term means may be indicated by an example or two. The rationals arranged in order of increasing magnitude constitute a special sort of series, a special sort of relation. Now consider two parts of this relation: (1) that part consisting of all the rationals equal to or less than the rational 2; (2) that part consisting of all the rationals whose squares (rationals) are less than the rational 2. Each of these parts is a relation, an ordered class of rationals, and is called a segment of rationals. Of these segments the first has a boundary, 2, but the second has no boundary. These segments are defined to be real numbers; the former one, the real (not the rational) number 2; the latter one, the real number, $\sqrt{2}$. In case of unbounded segments it has long been customary to "postulate" a boundary and to call it a real number. It involves some labor to show that the segments themselves, which do not require to be postulated, answer the purpose. And Mr. Russell's wit is justified: "The method of 'postulating' what we want has many advantages; they are the same as the advantages of theft over honest toil."

It is interesting to observe that in practice the same symbol is used to denote essentially different numbers. Thus, for example, the symbol, 2, is employed to denote a cardinal number, a positive integer, a rational number, a positive rational, a real number, and so on, all of which are radically different things.

For a fuller account of these matters and for a delightful excursion among kindred topics—complex and relation numbers, cardinal, ordinal and serial infinities, limits and continuity, selection theory, logical types, and so on—an interested reader must go to the book itself, and he will be amply rewarded whether he be a high school teacher of mathematics, a philosopher, or a research mathematician.

A distinguished British statesman has said: "I am a statesman but I am also a man." It is a pleasure to note, in closing this sketch, that Mr. Russell has occasionally relieved the austerity of his discourse by a brief and sudden sally, serving to show that the logical faculty, which is the human faculty *par* *excellence,* consists well with the warmer elements of a broader humanity. A citation or two may be in place. "Pure logic, and pure mathematics (which is the same thing), aims at being true, in Leibnizian phraseology, in all possible worlds, not only in this higgledy-piggledy job-lot of a world in which chance has imprisoned us." Again: "It is a disgrace to the human race that it has chosen to employ the same word 'is' in two such utterances as, Socrates is human, and Socrates is a man." Once more: "Men may be defined as featherless bipeds, or as rational animals, or (more correctly) by the traits by which Swift delineates the Yahoos."

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Elements of Vector Algebra. By L. SILBERSTEIN. London, Longmans, 1919. 8vo. 4 + 42 pp. Price $1.60.

Extract from the Preface—"This little book was written at the instance of Messrs. Adam Hilger, and, in accordance with their desire, it contains just what is required for the purpose of reading and handling my *Simplified Method of Tracing Rays, etc.* (Longmans, Green and Co., London, 1918). With this practical aim in view, all critical subleties have been purposely avoided. In fact, it is scarcely more than a synoptical presentation of the elements of vector algebra covering the needs of those engaged in geometrical optics. At the same time, however, it is hoped that this booklet will serve a more general purpose, viz., to provide everybody unacquainted with the subject with an easy introduction to the use of vector algebra.

"It is scarcely necessary to explain that the deductions given in this book are based on Euclid's axioms, notably with the inclusion of his postulate of parallels—upon which the equality of vectors is most essentially based. Those readers who are desirous of seeing how the formal rules here given can be generalized so as to be valid independently of the axioms of congruence and of parallels, may consult the author's *Projective Vector Algebra* (Bell and Sons, 1919), and a sequel to it published in *Phil. Mag.* for July, 1919, pp. 115–143. It is, however, advisable for the student to become first thoroughly familiar with the Euclidean vector algebra as here presented."

Contents—Section 1. Vectors defined, 1–2; 2. Equality of vectors defined, 2–3; 3. Addition of vectors, 3–10; 4. Subtraction of vectors, 10–11; 5. Scalar product of two vectors, 11–17; 6. The vector product of vectors, 17–21; 7. Expansion of vector formulæ, 21–23; 8. Iteration of vectorial multiplication 23–25; 9. The linear vector operator, 25–38; 10. Hints on differentiation of vectors, 38–40; Index, 41–42.

NOTES.

The Elementary Differential Geometry of Plane Curves by R. H. FOWLER, fellow of Trinity College, Cambridge is the latest issue, number 20, of the Cambridge Mathematical Tracts (Cambridge University Press, price 6 shillings).

The beautiful large pages, large type, and wide margins which the *American Journal of Mathematics* maintained for forty-one years have made way for pages about one-half as large, smaller type, and narrower margins.

Among the rich contents of *Archivio di Storia della Scienza,* volume 1, no. 2, June, 1919 (published February, 1920) are: an unpublished letter by Pierre Fermat, edited by G. Giovannozzi, 137–140; a bibliography of the printed works and of the facsimilies of manuscripts of Leonardo da Vinci, by A. Mieli, 177–187; a methodical bibliography of works on the history of science (326 titles) published in Italy since 1913, by A. Mieli, 195–217; and a review by G. Loria of L. C.